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On the Best Constant in the Error Bound for the H_0^1-Projection into Piecewise Polynomial Spaces

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Explicit *a priori* error bounds for the approximation by the H_0^1-projection into piecewise polynomial spaces are given. In particular, for the quadratic approximation, the optimal constant is derived, and a nearly optimal value for the cubic is obtained. These constants play an important role in the numerical verification method of finite element solutions for nonlinear elliptic equations.

1. INTRODUCTION

The purpose of this article is to bound the L^2-norm of the error in the least-squares approximation by piecewise polynomials in terms of the L^2-norm of the first derivative of the function being approximated. However, we prefer to describe this problem in the terms of the following application.

Let $u \in H_0^1(I) \cap H^2(I)$ be a function defined on the interval $I := [0, 1]$. We set a partition of I,

$$\Delta\colon 0 = x_0 < x_1 < \cdots < x_m = 1,$$

and the mesh size of this partition,

$$h := \max_{1 \leqslant i \leqslant m} (x_i - x_{i-1}).$$

For any nonnegative integer N, we define the piecewise polynomial space of degree $N+1$,

$$S_{\Delta, N}(I) := \{ p \in C(I) \mid p \text{ is a polynomial of degree} \leqslant N+1 \text{ on each subinterval } [x_{i-1}, x_i],\ 1 \leqslant i \leqslant m, \text{ with } p(0) = p(1) = 0 \}. \tag{1}$$

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We define the H_0^1-projection of u onto $S_{\Delta,N}(I)$ by

$$P_{\Delta,N}u \in S_{\Delta,N}(I), \qquad \text{such that} \quad ((P_{\Delta,N}u)', v') = (u', v'), \qquad \forall v \in S_{\Delta,N}(I),$$

where $(u, v) := [\int_0^1 u(x)\, v(x)\, dx]^{1/2}$ means the inner product in $L^2(I)$.

We consider error bounds for the H_0^1-projection of the form

$$\|u' - (P_{\Delta,N}u)'\| \leqslant Ch\, \|u''\|,$$

where $\|u\| := (u, u)^{1/2}$, and seek the smallest possible C, i.e., the number

$$C_N := \sup_{\Delta} \sup_{u \in H_0^1(I) \cap H^2(I),\, u \neq 0} \frac{\|u' - (P_{\Delta,N}u)'\|}{h\, \|u''\|}. \tag{2}$$

In particular, C_N depends only on N. When we are not able to determine C_N exactly, we look for good lower and upper bounds, i.e., for $\underline{C}_N$ and $\overline{C}_N$ with

$$\underline{C}_N \leqslant C_N \leqslant \overline{C}_N.$$

The explicit value of C_0 (i.e., for $N=0$, the piecewise linear case) equals $1/\pi$. Indeed, by the facts that the piecewise linear interpolation coincides with the H_0^1-projection and the error estimates in [8, Theorem 2.5], we have

$$\|u' - (P_{\Delta,0}u)'\| \leqslant \frac{h}{\pi}\, \|u''\|.$$

Equality holds iff $u(x) = \sin(\pi x/h)$ and Δ is a uniform partition. Therefore, $C_0 = 1/\pi$.

In this paper, we present the following results:

- We show that $C_1 = 1/(2\pi)$.
- We obtain the nearly optimal bounds $1/8.98954 \leqslant C_2 \leqslant 1/8.92338$ for the cubic case, $N=2$.

2. EXPLICIT ESTIMATES FOR C_N

2.1. *Reduction of the Problem*

As is well known, $u - P_{\Delta,N}u$ vanishes at x_i for all $0 \leqslant i \leqslant m$ (e.g., [8]) and, on each interval $[x_i, x_{i+1}]$, depends only on u on that interval. So we can reduce the problem of calculating C_N to each subinterval. Moreover, by a simple scaling, it is sufficient to consider only the case $m=1$. Therefore,

$$C_N = \sup_{u \in H_0^1(I) \cap H^2(I),\, u \neq 0} \frac{\|u' - (P_N u)'\|}{\|u''\|}, \tag{3}$$

with

$$P_N := P_{I,N}.$$

We also define another projection $\bar{u} \in Q_n(I) := S_{I,N}(I)$ as

$$\bar{u} = P_N \left[\sum_{k=1}^{N} \frac{(u, s_k)}{\|s_k\|^2} s_k \right], \tag{4}$$

with $s_k(x) := \sin(k\pi x)$, all k, and use it to determine a suitable constant $\bar{C}_N$ which satisfies

$$\|u' - (P_N u)'\| \leqslant \|u' - \bar{u}''\| \leqslant \bar{C}_N \|u''\|, \qquad \forall u \in H_0^1(I) \cap H^2(I), \tag{5}$$

as an upper bound for C_N.

2.2. *The Case of Polynomials of Degree* 2

We obtained the following theorem for $N=1$.

THEOREM 1. *In the quadratic case, the exact constant is* $C_1 = 1/(2\pi)$.

Proof. For any function $u \in H_0^1(I) \cap H^2(I)$, using the Fourier expansion of u,

$$u(x) \sim \sum_{k=1}^{\infty} u_k \sin(k\pi x),$$

the first derivative, $\bar{u}'$, of the function $\bar{u}$ defined in (4) can be written as

$$\bar{u}' = u_1(\pi c_1, L_1)\, L_1,$$

where, here and below, $c_k(x) := \cos(k\pi x)$, and L_i is the ith normalized Legendre polynomial defined by

$$L_i(x) := \frac{\sqrt{2i+1}}{i!} \frac{d^i}{dx^i} (x^i (1-x)^i).$$

Considering the orthogonality of the trigonometric functions, we have

$$\|u' - \bar{u}'\|^2 = \|u_1 \pi c_1\|^2 - \|u_1(\pi c_1, L_1)\, L_1\|^2 + \left\| \sum_{k=2}^{\infty} u_k k\pi c_k \right\|^2 - 2u_1(\pi c_1, L_1)\left(L_1, \sum_{k=2}^{\infty} u_k k\pi c_k \right). \tag{6}$$

From this, by Parseval's equality and the property of the H_0^1-projection,

$$\|u' - \bar{u}'\|^2 = \left(\frac{\pi^2}{2} - \frac{48}{\pi^2}\right) u_1{}^2 + \sum_{k=2}^{\infty} \frac{(k\pi)^2}{2} u_k{}^2 - \frac{96}{\pi^2} u_1 \sum_{l=2}^{\infty} \frac{u_{2l-1}}{2l-1}. \tag{7}$$

For arbitrary $\gamma_1 > 0$, this can be bounded as

$$\|u' - \bar{u}'\|^2 \leqslant \left(\frac{\pi^2}{2} - \frac{48}{\pi^2}\right) u_1{}^2 + \sum_{k=2}^{\infty} \frac{(k\pi)^2}{2} u_k{}^2 + \frac{48}{\pi^2}\left(\gamma_1 u_1{}^2 + \frac{1}{\gamma_1} \sum_{l=2}^{\infty} \left(\frac{u_{2l-1}}{2l-1}\right)^2 \right), \tag{8}$$

where we have used the inequality

$$2\,|ab| \leqslant \gamma_1 a^2 + \frac{1}{\gamma_1} b^2 \qquad (\forall \gamma_1 > 0).$$

Moreover, by the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\left(\sum_{l=2}^{\infty} \frac{u_{2l-1}}{2l-1} \right)^2 &= \left(\sum_{l=2}^{\infty} \frac{1}{(2l-1)^3} \cdot (2l-1)^2\, u_{2l-1} \right)^2 \\
&\leqslant \left(\sum_{l=2}^{\infty} \frac{1}{(2l-1)^6} \right)\left(\sum_{l=2}^{\infty} (2l-1)^4\, u_{2l-1}{}^2 \right) \\
&= \left(\frac{(2^6-1)\,\pi^6}{2 \cdot 6!} B_6 - 1 \right)\left(\sum_{l=2}^{\infty} (2l-1)^4\, u_{2l-1}{}^2 \right) \\
&= \left(\frac{\pi^6}{960} - 1 \right)\left(\sum_{l=2}^{\infty} (2l-1)^4\, u_{2l-1}{}^2 \right),
\end{aligned} \tag{9}$$

where B_n is the Bernoulli number.

Thus, from (6)–(9), we obtain

$$\begin{aligned}
\|u' - \bar{u}'\|^2 &\leqslant \left(\frac{\pi^2}{2} - \frac{48}{\pi^2} + \frac{48}{\pi^2} \gamma_1 \right) u_1{}^2 \\
&\quad + \sum_{l=2}^{\infty} \left[\frac{((2l-1)\,\pi)^2}{2} + \frac{48}{\pi^2}\left(\frac{\pi^6}{960} - 1 \right)(2l-1)^4 \frac{1}{\gamma_1} \right] u_{2l-1}{}^2 \\
&\quad + \sum_{l=1}^{\infty} \frac{(2l\pi)^2}{2} u_{2l}{}^2.
\end{aligned} \tag{10}$$

Now observe that

$$\|u''\|^2 = \sum_{k=1}^{\infty} \frac{(k\pi)^4}{2} u_k{}^2, \tag{11}$$

from the Fourier expansion of u'' and Parseval's equality. By comparing the corresponding coefficients of each u_k^2 in (10) and (11), we get the following sufficient conditions for $\bar{C}_1$ in (5):

(i) $$\frac{\pi^2}{2} - \frac{48}{\pi^2} + \frac{48}{\pi^2} \gamma_1 \leqslant \frac{\pi^4}{2} \bar{C}_1^2,$$

(ii) $$\frac{((2l-1)\pi)^2}{2} + \frac{48}{\pi^2}\left(\frac{\pi^6}{960} - 1\right)(2l-1)^4 \frac{1}{\gamma_1} \leqslant \frac{((2l-1)\pi)^4}{2} \bar{C}_1^2 \qquad (\forall l \geqslant 2),$$

(iii) $$\frac{1}{(2l\pi)^2} \leqslant \bar{C}_1^2 \qquad (\forall l \geqslant 1).$$

Condition (i) is equivalent to

$$\text{(i)}' \quad \gamma_1 \leqslant \frac{\pi^4}{96}(\bar{C}_1^2 \pi^2 - 1) + 1.$$

Condition (ii) is equivalent to

$$\frac{48}{\pi^2 \gamma_1}\left(\frac{\pi^6}{960} - 1\right) \leqslant \frac{\pi^2}{2}\left(\bar{C}_1^2 \pi^2 - \frac{1}{(2l-1)^2}\right) \qquad (\forall l \geqslant 2),$$

and this can be reduced to the inequality for the smallest l, i.e., $l = 2$,

$$\frac{48}{\pi^2 \gamma_1}\left(\frac{\pi^6}{960} - 1\right) \leqslant \frac{\pi^2}{2}\left(\bar{C}_1^2 \pi^2 - \frac{1}{9}\right).$$

So we have the following condition equivalent to (ii):

$$\text{(ii)}' \quad \frac{96}{\pi^4}\left(\frac{\pi^6}{960} - 1\right)\left(\bar{C}_1^2 \pi^2 - \frac{1}{9}\right)^{-1} \leqslant \gamma_1.$$

By (i)′ and (ii)′, the existence of $\gamma_1 > 0$ is equivalent to

$$\frac{\pi^6}{10} - 96 \leqslant \pi^4 \left(\frac{\pi^4}{96}(\pi^2 \bar{C}_1^2 - 1) + 1\right)\left(\pi^2 \bar{C}_1^2 - \frac{1}{9}\right). \tag{12}$$

It is easily seen that the smallest $\bar{C}_1$ satisfying this inequality is smaller than $1/(2\pi)$ which is the minimal solution of (iii). Thus, we obtain (5) with $\bar{C}_1 = 1/(2\pi)$ for this case. On the other hand, for $u = s_2$, we have $u' = 2\pi c_2$, while $(P_1 u)' = 0$ and $u'' = -4\pi^2 s_2$, hence $\|u' - (P_1 u)'\| / \|u''\| = 1/(2\pi)$. Thus, $C_1 = \bar{C}_1 = 1/(2\pi)$. ∎

2.3. *The Case of Polynomials of Degree* 3

It is tempting to deduce from the cases $N = 0$ and $N = 1$ that, for any N,

$$C_N = \frac{1}{(N+1)\,\pi}.$$

But, already for $N = 2$, we found the counterexample

$$\underline{u}(x) := \sin\left(\frac{\pi x}{h}\right)^3, \qquad \frac{1}{3\pi} < \frac{\|\underline{u}' - (P_2 \underline{u})'\|}{h\,\|\underline{u}''\|} \approx \frac{1}{8.98954}. \tag{13}$$

Then, we adopt the right-hand side as $\underline{C}_2$.

As to an upper bound, $\bar{C}_2$, we have the following theorem.

THEOREM 2. *For any* $u \in H_0^1(I) \cap H^2(I)$ *and its* H_0^1*-projection* $P_2 u$ *into* $Q_2(I)$,

$$\|u' - (P_2 u)'\| \leqslant \bar{C}_2\,\|u''\|, \tag{14}$$

with

$$\begin{aligned}\bar{C}_2 :=& \frac{1}{3\sqrt{5}\,\pi^3}\\ &\times(-2160 + 25\pi^4 + 4\sqrt{5}\sqrt{-174960 - 1080\pi^4 + 243\pi^6 + 5\pi^8})^{1/2}\\ \approx& \frac{1}{8.92337}.\end{aligned}$$

Proof. First note that, for $\bar{u}$ given in (4), $\bar{u}'$ can be written as

$$\bar{u}' = u_1(\pi c_1, L_1)\,L_1 + u_2(2\pi c_2, L_2)\,L_2.$$

Considering the orthogonality of the trigonometric functions, Parseval's equality, and the property of H_0^1-projection, we have

$$\begin{aligned}
\|u' - \bar{u}'\|^2 = {} & \|u_1 \pi c_1\|^2 - \|u_1(\pi c_1, L_1) L_1\|^2 \\
& + \|u_2 2\pi c_2\|^2 - \|u_2(2\pi c_2, L_2) L_2\|^2 + \left\| \sum_{k=3}^{\infty} u_k k\pi c_k \right\|^2 \\
& - 2u_1(\pi c_1, L_1) \left(L_1, \sum_{k=3}^{\infty} u_k k\pi c_k \right) \\
& - 2u_2(2\pi c_2, L_2) \left(L_2, \sum_{k=3}^{\infty} u_k k\pi c_k \right) \\
= {} & \left(\frac{\pi^2}{2} - \frac{48}{\pi^2} \right) u_1^2 + \left(2\pi^2 - \frac{180}{\pi^2} \right) u_2^2 \\
& + \sum_{k=3}^{\infty} \frac{(k\pi)^2}{2} u_k^2 - 2 \cdot \frac{48}{\pi^2} u_1 \sum_{l=2}^{\infty} \frac{u_{2l-1}}{2l-1} \\
& - 2 \cdot \frac{180}{\pi^2} u_2 \sum_{l=2}^{\infty} \frac{u_{2l}}{2l},
\end{aligned} \tag{15}$$

Arguing as in the proof for Theorem 1 and with the equality

$$\sum_{l=2}^{\infty} \frac{1}{l^6} = \frac{(2\pi)^6}{2 \cdot 6!} B_6 - 1 = \frac{\pi^6}{945} - 1,$$

we have, for arbitrary $\gamma_1 > 0$ and $\gamma_2 > 0$,

$$\begin{aligned}
\|u' - \bar{u}'\|^2 \leqslant {} & \left(\frac{\pi^2}{2} - \frac{48}{\pi^2} + \frac{48}{\pi^2} \gamma_1 \right) u_1^2 \\
& + \sum_{l=2}^{\infty} \left[\frac{((2l-1)\,\pi)^2}{2} + \frac{48}{\pi^2} \left(\frac{\pi^6}{960} - 1 \right) (2l-1)^4 \frac{1}{\gamma_1} \right] u_{2l-1}^2 \\
& + \left(\frac{(2\pi)^2}{2} - \frac{180}{\pi^2} + \frac{180}{\pi^2} \gamma_2 \right) u_2^2 \\
& + \sum_{l=2}^{\infty} \left[\frac{((2l)\,\pi)^2}{2} + \frac{180}{\pi^2} \frac{1}{16} \left(\frac{\pi^6}{945} - 1 \right) (2l)^4 \frac{1}{\gamma_2} \right] u_{2l}^2.
\end{aligned} \tag{16}$$

By comparing the corresponding coefficients in the above and (11), we obtain sufficient conditions for $\overline{C}_2$ in (5):

$$\text{(I)}\quad \frac{\pi^2}{2} - \frac{48}{\pi^2} + \frac{48}{\pi^2}\gamma_1 \leqslant \frac{\pi^4}{2}\bar{C}_2^2 \qquad \text{(coefficients of } u_1^2\text{)},$$

$$\text{(II)}\quad \frac{((2l-1)\pi)^2}{2} + \frac{48}{\pi^2}\left(\frac{\pi^6}{960} - 1\right)(2l-1)^4 \frac{1}{\gamma_1} \leqslant \frac{((2l-1)\pi)^4}{2}\bar{C}_2^2$$

$$(\forall l \geqslant 2) \text{ (coefficients of } u_k^2 \text{ for odd integers } k \geqslant 3),$$

$$\text{(III)}\quad \frac{(2\pi)^2}{2} - \frac{180}{\pi^2} + \frac{180}{\pi^2}\gamma_2 \leqslant \frac{(2\pi)^4}{2}\bar{C}_2^2 \qquad \text{(coefficients of } u_2^2\text{)},$$

$$\text{(IV)}\quad \frac{(2l\pi)^2}{2} + \frac{180}{\pi^2}\frac{1}{16}\left(\frac{\pi^6}{945} - 1\right)(2l)^4 \frac{1}{\gamma_2} \leqslant \frac{(2l\pi)^4}{2}\bar{C}_2^2$$

$$(\forall l \geqslant 2) \text{ (coefficients of } u_k^2 \text{ for even integers } k \geqslant 4).$$

We now try to find the smallest $\bar{C}_2 > 0$ for which there exists $\gamma_1 > 0$ and $\gamma_2 > 0$ satisfying (I)–(IV). This problem is reduced to getting the smallest solution of the following three inequalities for $\bar{C}_2^2$ (the first, from (I) and (II), is just (12), the second follows from (III) and (IV), the third by (13)):

$$\frac{\pi^6}{10} - 96 \leqslant \pi^4\left(\frac{\pi^4}{96}(\pi^2\bar{C}_2^2 - 1) + 1\right)\left(\pi^2\bar{C}_2^2 - \frac{1}{9}\right)$$

$$\frac{\pi^6}{84} - \frac{45}{4} \leqslant \pi^4\left(\frac{\pi^4}{180}(4\pi^2\bar{C}_2^2 - 1) + \frac{1}{2}\right)\left(\pi^2\bar{C}_2^2 - \frac{1}{16}\right)$$

$$\frac{1}{3\pi} < \bar{C}_2. \tag{17}$$

Straightforward calculations of the solution range of each of these inequalities show that the desired minimal solution $\bar{C}_2$ of (I)–(IV) is provided by the larger of the two places of equality in the first inequality in (17). This is given by

$$\bar{C}_2 := \frac{1}{3\sqrt{5}\,\pi^3} \times (-2160 + 25\pi^4 + 4\sqrt{5}\sqrt{-174960 - 1080\pi^4 + 243\pi^6 + 5\pi^8})^{1/2}, \tag{18}$$

and thus Theorem 2 is proved. ∎

3. AN APPLICATION: THE MULTI-DIMENSIONAL CASE

In the present section, we show that our result can be extended to the multidimensional case that involves the tensor product of one-dimensional piecewise polynomial spaces.

Let Ω be a bounded convex domain in $\mathbf{R}^n$. $(\cdot, \cdot)_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$ denote the inner product and the associated norm in $L^2(\Omega)$, respectively. We use several Sobolev spaces such as $H^1(\Omega)$, $H_0^1(\Omega)$, and so on. In particular,

$$|\psi|_{H^2(\Omega)} := \left[\sum_{|\alpha|=2} \|D^\alpha \psi\|_{L^2(\Omega)}^2 \right]^{1/2}$$

denotes the semi-norm of $H^2(\Omega)$ where

$$\alpha =: (\alpha_1, \alpha_2, ..., \alpha_n), \qquad \alpha_j \in \mathbf{N},$$

$$|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

$$D^\alpha \psi := \frac{\partial^{|\alpha|} \psi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

We only describe an example for the case of the region $\Omega = [0, 1] \times [0, 1]$, but the higher dimensional extension will be straightforward. We consider the tensor product space of the one dimensional piecewise polynomials of degree $\leqslant N+1$. Since the H_0^1-projection Pu of $u \in H_0^1(\Omega)$ into the approximation space $S(\Omega) := S_{\Delta, N}(I) \otimes S_{\Delta, N}(I)$ satisfies

$$\|\nabla(u - Pu)\|_{L^2(\Omega)} \leqslant \|\nabla(u - v)\|_{L^2(\Omega)} \qquad \text{for} \quad \forall v \in S(\Omega), \tag{19}$$

we obtain the inequality

$$\|\nabla(u - Pu)\|_{L^2(\Omega)} \leqslant \|\nabla(u - P_y P_x u)\|_{L^2(\Omega)}. \tag{20}$$

Here, $P_x u(\cdot, y)$ means the one dimensional H_0^1-projection in x for fixed y, and $P_y(x, \cdot)$ as well.

Then, by (2), we have

$$\|u - P_x u\|_{L^2(\Omega)} \leqslant C_N h \, \|u_x\|_{L^2(\Omega)}.$$

Therefore, by the orthogonality of the H_0^1-projection, we get

$$\begin{aligned}
\|(u - P_y P_x u)_x\|_{L^2(\Omega)}^2 &= \|(u - P_x u)_x\|_{L^2(\Omega)}^2 + \|(P_x u - P_y P_x u)_x\|_{L^2(\Omega)}^2 \\
&= \|(u - P_x u)_x\|_{L^2(\Omega)}^2 + \|(P_x(u - P_y u))_x\|_{L^2(\Omega)}^2 \\
&\leqslant \|(u - P_x u)_x\|_{L^2(\Omega)}^2 = \|(u - P_y u)_x\|_{L^2(\Omega)}^2 \\
&\leqslant C_N^2 h^2 (\|u_{xx}\|_{L^2(\Omega)}^2 + \|u_{xy}\|_{L^2(\Omega)}^2).
\end{aligned} \tag{21}$$

Similarly, we have

$$\|(u-P_xP_yu)_y\|^2_{L^2(\Omega)} \leqslant C_N^2h^2(\|u_{yy}\|^2_{L^2(\Omega)}+\|u_{yx}\|^2_{L^2(\Omega)}). \tag{22}$$

Thus, we obtain the estimate

$$\|\nabla(u-Pu)\|_{L^2(\Omega)} \leqslant C_Nh\,|u|_{H^2(\Omega)}, \tag{23}$$

which implies that the same constant as in the one-dimensional case can also be used for two dimensions.

Remark. In the numerical verification methods for nonlinear elliptic boundary value problems, the magnitude of the constants in the *a priori* error estimates for finite element solutions of Poisson's equation plays an essential role [6, 7, 9, 10]. In general, these constants have to be estimated as sharply as possible, because their size seriously affects the efficiency of verification cost. This is the principal motivation of our present work. The practical and efficient estimates of such constants for a triangular finite element mesh would also be an important task in the future.

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